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Higher order matrix SUSY transformations in two-dimensional quantum mechanics

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Abstract

The iteration procedure of supersymmetric transformations for the two-dimensional Schrödinger operator is implemented by means of the matrix form of factorization in terms of matrix 2×2 supercharges. Two different types of iterations are investigated in detail. The particular case of diagonal initial Hamiltonian is considered, and the existence of solutions is demonstrated. Explicit examples illustrate the construction.

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1. Introduction

Supersymmetric quantum mechanics (SUSY QM) [1, 2] has become a well-known powerful tool for the investigation of problems of modern non-relativistic quantum mechanics. The main achievements of SUSY QM during the last two decades involved isospectral pairs of one-dimensional systems. In particular, the conventional SUSY QM (with supercharges of first order in derivatives) was generalized [3, 4] to higher order SUSY QM (HSUSY QM) with supercharges nth order polynomials in momentum. This generalization induces the deformation of SUSY QM algebra: the anticommutator of supercharges is now a polynomial in the super-Hamiltonian. The class of pairs of isospectral systems which are connected (intertwined) by components of supercharge is thus enlarged [4]. Two classes of second-order supercharges were found: *reducible*, which can be written as a chain of consecutive first-order transformations, and *irreducible* second-order transformations. In both cases the intertwined Hamiltonians are expressed in terms of only one arbitrary function [4]. For a (admittedly nonexhaustive) list of references devoted to one-dimensional HSUSY QM see, in particular, [5].

It is also useful to recall the equivalent terminology related to the classic mathematical paper by Darboux [6]. The conventional first-order SUSY transformations coincide with the so-called Darboux transformations, which are very useful in mathematical physics (see, for example, [7] for nonlinear evolution equations). Then nth order HSUSY transformations correspond to the well-known Crum–Krein transformations [8], where the eigenfunctions of nth iterated Hamiltonian are obtained in terms of n given eigenfunctions of the initial Hamiltonian by means of 'determinant expressions'. The second iterations of SUSY transformations for matrix one-dimensional potentials were considered in [9], recently the general form of matrix Crum–Krein transformations was constructed [10].

Much less attention was paid in the literature to SUSY QM of two-dimensional systems. Here we list some partial achievements which have been obtained in this field. The direct multi-dimensional generalization of the standard one-dimensional SUSY QM was elaborated in [11, 12] for arbitrary space dimensions. In the simplest two-dimensional case the supersymmetrization of a given (scalar) Schrödinger Hamiltonian involves not only a second scalar Hamiltonian but also a matrix 2×2 Schrödinger operator. The spectra of these three components of the super-Hamiltonian are interrelated, and their wavefunctions are connected by the components of supercharges. This approach was used successfully [13, 14] to find the spectrum and the wavefunctions of Pauli operators describing spin $1/2$ particles (with arbitrary values of the gyromagnetic ratio) in a broad class of non-homogeneous external electromagnetic fields in terms of spectra and eigenfunctions of two scalar Hamiltonians.

In the series of papers [15] direct intertwining relations between a pair of scalar two-dimensional Hamiltonians were studied. A wide variety of particular solutions was found, leading to new integrable quantum systems which are not amenable to separation of variables. This approach allowed us to introduce [16] two new methods for the study of the two-dimensional Schrödinger equation: *SUSY*-separation of variables and shape invariance. Multi-dimensional SUSY QM for an arbitraryly oriented Riemannian manifolds was studied in [17]. In a very recent paper [18] the Hamilton–Jacobi approach of classical mechanics was applied to study classical two-dimensional supersymmetric models and their quantization.

The present paper addresses the problem of the construction of a chain of consecutive supersymmetric (generalized Darboux) transformations for two-dimensional Schrödinger systems allowing for a deeper insight into isospectrality in multi-dimensional cases. Let us remark that the only early attempt to construct such iterations failed (see section 4 of the first paper in [11]). Even the very possibility of making iterations for two-dimensional systems remained a task to be solved.

The new matrix supersymmetric factorization of the two-dimensional Pauli operator, recently proposed in [14], can provide a new chance to construct iterations of first-order (reducible) supersymmetric transformations. Indeed, in contrast to the method of quasifactorization [11], where the vector components of supercharges include scalar superpotential, the matrix factorization of [14] involves the components of supercharges with matrix superpotentials. Therefore all elements of the SUSY intertwining relations are now represented by 2×2 matrix differential operators.

The organization of the paper is as follows. In section 2 the chain of SUSY transformations is performed by gluing the corresponding components of two successive super-Hamiltonians. The generalization of this prescription includes an additional global unitary transformation (section 3). In section 4 we analyse the particularly interesting case when the initial Hamiltonian is diagonal, and one supersymmetrizes a scalar Hamiltonian by building partner Hamiltonians via second-order SUSY transformations. This construction, in reverse order, can also be used for diagonalization of a class of two-dimensional 2×2 matrix Schrödinger operators or for matrix systems of coupled second-order differential equations.

2. Second iteration by gluing

Before introducing the new form of SUSY factorization for two-dimensional systems, proposed in [14], let us start from the main formulae for the conventional [2] one-dimensional SUSY QM with partner Hamiltonians and their eigenfunctions:

$$H^{(0)} = q^+q^- = -\partial^2 + V^{(0)}(x) \qquad H^{(0)}\Psi_n^{(0)}(x) = E_n\Psi_n^{(0)}(x) \tag{1}$$

$$H^{(1)} = q^-q^+ = -\partial^2 + V^{(1)}(x) \qquad H^{(1)}\Psi_n^{(1)}(x) = E_n\Psi_n^{(1)}(x). \tag{2}$$

The operators $q^{\pm}$ are defined as

$$q^+ = -\partial + W(x) \qquad q^- = (q^+)^\dagger = +\partial + W(x) \qquad \partial \equiv \mathrm{d}/\mathrm{d}x$$

and they intertwine the components (1), (2) of the super-Hamiltonian:

$$H^{(0)}q^+ = q^+H^{(1)} \qquad q^-H^{(0)} = H^{(1)}q^-. \tag{3}$$

Then, up to normalization factors,

$$\Psi_n^{(1)}(x) = q^-\Psi_n^{(0)}(x) \qquad \Psi_n^{(0)}(x) = q^+\Psi_n^{(1)}(x). \tag{4}$$

The intertwinings (3) are the most important relations of the SUSY QM algebra:

$$\{\hat{Q}^+, \hat{Q}^-\} = \hat{H} \qquad (\hat{Q}^+)^2 = (\hat{Q}^-)^2 = 0 \qquad [\hat{H}, \hat{Q}^{\pm}] = 0 \tag{5}$$

where the super-Hamiltonian $\hat{H}$ and supercharges $\hat{Q}^{\pm}$ are

$$\hat{H} = \begin{pmatrix} H^{(0)} & 0 \\ 0 & H^{(1)} \end{pmatrix} \qquad \hat{Q}^+ = (\hat{Q}^-)^\dagger = \begin{pmatrix} 0 & 0 \\ q^- & 0 \end{pmatrix}. \tag{6}$$

This procedure of supersymmetric (Darboux) transformations from $H^{(0)}, \Psi_n^{(0)}(x), E_n$ to $H^{(1)}, \Psi_n^{(1)}(x), E_n$ was iterated [4] by reducible supercharges of second order in derivatives. This iteration leads to a polynomial deformation of SUSY algebra.

A two-dimensional generalization of standard SUSY QM [11] organized (in the super-Hamiltonian) one 2×2 matrix and two scalar Hamiltonians[5]. Each scalar Hamiltonian is separately intertwined with the matrix Hamiltonian, but the spectra of the two scalar Hamiltonians are not related. This construction was based on the quasifactorization of all three components of the super-Hamiltonian in terms of the first-order differential operators $q_l^{\pm} = \mp\partial_l + (\partial_l\chi(x))$ and[6] $p_l^{\pm} = \epsilon_{lk}q_k^{\mp}; l, k = 1, 2$. A similar quasifactorization with new supercharges, used in [11] to build an iteration of SUSY transformations, turned out to be unsuccessful since several necessary constraints were too complicated to be implemented [11]. In addition, it led to a growth of the matrix dimensionality of the new Hamiltonians.

In the present paper, we use the method [14] based on the matrix factorization of components of the two-dimensional Hermitian super-Hamiltonian[7]:

$$H^{(0)} = q^+q^- = -\Delta^{(2)} + (W_i)^2 + \sigma_1(\partial_2W_1 - \partial_1W_2) - \sigma_3(\partial_1W_1 + \partial_2W_2) \tag{7}$$

[5] For the case of an arbitrary dimensionality d of coordinate space the super-Hamiltonian contains $(d+1)$ components $H^{(n)}, n = 0, 1, \ldots, d$ with *matrix* dimensionality C_d^n, i.e. dimensionality of the space with definite fermionic occupation number n (see details in [12]).

[6] The summation over repeated indices ($i = 1, 2$) is implied here and below.

[7] Slightly different notation is chosen here in comparison with [14].

$$H^{(1)} = q^- q^+ = -\Delta^{(2)} + (W_i)^2 + \sigma_1(\partial_2 W_1 + \partial_1 W_2) + \sigma_3(\partial_1 W_1 - \partial_2 W_2)$$
$$\partial_i \equiv \frac{\partial}{\partial x_i} \qquad (i = 1, 2) \qquad \Delta \equiv \partial_1^2 + \partial_2^2. \tag{8}$$

The 2×2 components $q^\pm$ of 4×4 supercharge $\hat{Q}^\pm$ (see equation (6)) are matrix differential operators of first order in derivatives[8],

$$q^\pm = \mp\partial_1 - \mathrm{i}\sigma_2\partial_2 + \sigma_1 W_2 + \sigma_3 W_1 \qquad q^+ = (q^-)^\dagger \tag{9}$$

where $W_{1,2}$ are real, and σ_i $(i = 1, 2, 3)$ are standard Pauli matrices. By construction, these operators intertwine the Hamiltonians (7), (8) according to the standard relations (3) of SUSY algebra. Relations (4)–(6) of one-dimensional SUSY QM are satisfied as well, with the proviso that we deal now with 2×2 *matrix* differential operators $q^\pm$, $H^{(0)}$, $H^{(1)}$ and two-component wavefunctions $\Psi_n^{(0)}, \Psi_n^{(1)}$. Isospectrality of the (matrix) Hamiltonians $H^{(0)}, H^{(1)}$ holds, as usual, except for the possible zero modes of $q^\pm$.

To iterate supersymmetric transformations we impose the ladder equation (see [3, 4]) thus gluing the lower component $H^{(1)}$ of the super-Hamiltonian $\hat{H}$ with the upper component $\widetilde{H}^{(0)}$ of the next super-Hamiltonian $\hat{\widetilde{H}}$, defined by (7), (8) with W replaced by $\widetilde{W}$:

$$H^{(1)} = \widetilde{H}^{(0)} + C \qquad C = \text{const (real)}. \tag{10}$$

Then the second-order supercharge $Q^+ \equiv q^+\tilde{q}^+$ gives the following intertwining relations:

$$H^{(0)}Q^+ = Q^+(\widetilde{H}^{(1)} + C).$$

The conditions (10) can be rewritten as a nonlinear system of differential equations:

$$W_i^2 = \widetilde{W}_i^2 + C \tag{11}$$

$$\partial_2 W_1 + \partial_1 W_2 = \partial_2 \widetilde{W}_1 - \partial_1 \widetilde{W}_2 \tag{12}$$

$$\partial_1 W_1 - \partial_2 W_2 = -\partial_1 \widetilde{W}_1 - \partial_2 \widetilde{W}_2. \tag{13}$$

It may be convenient to express these ladder equations in terms of a complex function of two mutually conjugated complex variables

$$W(z, z^\star) \equiv W_1(z, z^\star) + \mathrm{i}W_2(z, z^\star) \qquad z \equiv x_1 + \mathrm{i}x_2 \quad z^\star \equiv x_1 - \mathrm{i}x_2 \tag{14}$$

as follows:

$$|W(z, z^\star)|^2 = |\widetilde{W}(z, z^\star)|^2 + C \tag{15}$$

$$\bar\partial W(z, z^\star) = -\partial\widetilde{W}(z, z^\star)$$
$$\partial \equiv \frac{\partial}{\partial z} = \frac{1}{2}(\partial_1 - \mathrm{i}\partial_2) \qquad \bar\partial \equiv \frac{\partial}{\partial z^\star} = \frac{1}{2}(\partial_1 + \mathrm{i}\partial_2). \tag{16}$$

The general solution of equation (16) can be written by means of an *arbitrary* complex function,

$$W(z, z^\star) = -\partial F(z, z^\star) \qquad \widetilde{W}(z, z^\star) = \bar\partial F(z, z^\star) \qquad F(z, z^\star) \equiv F_1(z, z^\star) + \mathrm{i}F_2(z, z^\star) \tag{17}$$

with real F_1 and F_2.

In terms of $F_1(z, z^\star)$, $F_2(z, z^\star)$ equation (15) reads

$$(\partial_1 F_1)(\partial_2 F_2) - (\partial_2 F_1)(\partial_1 F_2) = C. \tag{18}$$

[8] An analogous form of matrix supercharges was used also in [19] in the context of ParaSUSY QM.

Its general solution is a general solution of the homogeneous equation summed to a particular solution of the inhomogeneous one. One can check that the general solution of the homogeneous equation requires that $F_2(x_1, x_2)$ depend on its arguments via $F_1(x_1, x_2)$ only

$$F_2(x_1, x_2) = \Phi[F_1(x_1, x_2)] \qquad F(z, z^\star) = F_1(x_1, x_2) + \mathrm{i}F_2(x_1, x_2) \tag{19}$$

(and, of course, vice versa). In order to find the particular solution of inhomogeneous equation (18) one has to introduce new variables y_1, y_2 such that $y_2 \equiv x_2$ and y_1 coincides with one of the solutions (19), e.g. $y_1 \equiv F_1(x_1, x_2)$. In new variables equation (18) becomes

$$\frac{\partial F_1(x_1, x_2)}{\partial x_1}\frac{\partial F_2(y_1, y_2)}{\partial y_2} = C$$

and its solution for F_2 in terms of F_1 is

$$F_2(x_1, x_2) = C\int \mathrm{d}y_2 \left(\frac{\partial F_1(x_1, x_2)}{\partial x_1}\right)^{-1}$$

where the partial derivative is taken with $x_2 = \text{const}$, and after integration over y_2 the initial variables x_1, x_2 are to be re-inserted. So the general solution of (18) is the sum

$$F_2(x_1, x_2) = \Phi[F_1(x_1, x_2)] + C\int \mathrm{d}y_2 \left(\frac{\partial F_1(x_1, x_2)}{\partial x_1}\right)^{-1}. \tag{20}$$

Thus two matrix two-dimensional Hamiltonians (components of new super-Hamiltonian) $H^{(0)}$ (see expression (7)) and $(\widetilde{H}^{(1)} + C)$, which can be written as

$$\widetilde{H}^{(1)} + C = H^{(0)} + 2\sigma_1\partial_1\partial_2F_1(x_1, x_2) - 2\sigma_3\partial_1\partial_2F_2(x_1, x_2)$$

are intertwined by the differential matrix operator (component of new supercharges):

$$\begin{aligned} Q^+ \equiv q^+\tilde{q}^+ = {} & \partial_1^2 - \partial_2^2 + 2\mathrm{i}\sigma_2\partial_1\partial_2 + \frac{1}{4}[(\partial_2F_2)^2 + (\partial_2F_1)^2 - (\partial_1F_2)^2 - (\partial_1F_1)^2] \\ & + \sigma_1\left[(\partial_1F_1)\partial_2 - (\partial_2F_1)\partial_1 - \frac{1}{2}\left(\partial_1^2 + \partial_2^2\right)F_2\right] \\ & - \frac{\mathrm{i}}{2}\sigma_2[(\partial_1F_1)(\partial_2F_1) + (\partial_1F_2)(\partial_2F_2)] \\ & + \sigma_3\left[(\partial_2F_2)\partial_1 - (\partial_1F_2)\partial_2 + \frac{1}{2}\left(\partial_1^2 + \partial_2^2\right)F_1\partial_1\right]. \end{aligned}$$

The SUSY algebra has now a polynomial form:

$$\{\hat{Q}^+, \hat{Q}^-\} = \hat{H}(\hat{H} - C) \qquad (\hat{Q}^+)^2 = (\hat{Q}^-)^2 = 0 \qquad [\hat{H}, \hat{Q}^\pm] = 0.$$

This algebra differs essentially from the deformed SUSY algebra for a class of two-dimensional systems investigated in a series of papers [15] in so far as it *does not* incorporate any non-trivial symmetry operators (no central extension): thus we have no positive indications of integrability of the systems $H^{(0)}, \widetilde{H}^{(1)}$.

Higher order iterations. In order to perform the next iteration of supersymmetric (Darboux) transformations one has to glue $\widetilde{H}^{(1)}$ with the first component $\widetilde{\widetilde{H}}^{(0)}$ of a new super-Hamiltonian. This procedure follows the previous one: superpotentials are expressed in terms of derivatives of the same arbitrary complex function $\widetilde{F}(z, z^\star)$:

$$\widetilde{W} = \widetilde{W}_1 + \mathrm{i}\widetilde{W}_2 = +\bar{\partial}F(z, z^\star) = -\partial\widetilde{F} \tag{21}$$

$$\widetilde{\widetilde{W}} = \widetilde{\widetilde{W}}_1 + \mathrm{i}\widetilde{\widetilde{W}}_2 = +\bar{\partial}\widetilde{F}(z, z^\star). \tag{22}$$

The last equality in (21) implies that in terms of an arbitrary function $f(z, z^\star)$:

$$F(z, z^\star) = -\partial f(z, z^\star) \qquad \widetilde{F}(z, z^\star) = +\bar{\partial} f(z, z^\star).$$

It is difficult to find the general solution of the nonlinear equations (15) and the analogous one for $\widetilde{W}$, $\widetilde{\widetilde{W}}$, but it is possible to find particular solutions with certain simplifying ansätze. For example, explicit solutions can be obtained for $C = \widetilde{C} = 0$ and $f(z, z^\star) \equiv |\phi(z)|^2$.

3. Iterations by gluing and global transformations

We now modify the procedure (namely, equation (10)) of the previous section by gluing the Hamiltonians $H^{(1)}$ and $\widetilde{H}^{(0)}$ *up to a global unitary transformation* U. Such a procedure allows us to preserve the main details of the scheme, but it might extend the class of intertwined Hamiltonians.

Without loss of generality, we may restrict ourselves[9] to $U = \mathrm{i}\sigma_3$:

$$H^{(1)} = U\widetilde{H}^{(0)}U^{-1} + C \qquad U = \mathrm{i}\sigma_3. \tag{23}$$

In this case (15), (16) must be replaced by

$$|W(z, z^\star)|^2 = |\widetilde{W}(z, z^\star)|^2 + C \tag{24}$$

$$\partial_1(\widetilde{W}_2 - W_2) = \partial_2(W_1 + \widetilde{W}_1) \tag{25}$$

$$\partial_1(\widetilde{W}_1 + W_1) = \partial_2(W_2 - \widetilde{W}_2). \tag{26}$$

The equations (25) and (26) can be solved explicitly: W and $\widetilde{W}$ are interrelated by an arbitrary *holomorphic function* $G(z)$:

$$W(z, z^\star) + \widetilde{W}^\star(z, z^\star) = G(z) \qquad G(z) \equiv G_1 + \mathrm{i}G_2 \qquad \partial_1 G_1 = \partial_2 G_2 \\ \partial_1 G_2 = -\partial_2 G_1. \tag{27}$$

The remaining nonlinear equation (24) for G can be written as:

$$G_1^2 + G_2^2 - 2W_2G_2 - 2W_1G_1 + C = 0. \tag{28}$$

It can be solved explicitly as a linear *algebraic equation* for superpotentials W_1, W_2 in terms of an arbitrary holomorphic function $G(z)$. For an arbitrary $G(z)$ one can choose also an arbitrary function $W_2(z, z^\star)$ and find the functions $W_1(z, z^\star)$, $\widetilde{W}_{1,2}(z, z^\star)$ from equations (28) and (27). Thus one obtains a class of Hamiltonians (7) for which the second-order matrix supersymmetric transformations can be performed. The resulting Hamiltonian

$$\widetilde{H}^{(1)} + C = H^{(0)} + 2\sigma_1(\partial_1 W_2 - \partial_2 W_1 + \partial_2 G_1) + 2\sigma_3\partial_1 G_1 \tag{29}$$

is intertwined with the initial Hamiltonian $H^{(0)}$ by the operator

$$Q^+ = q^+\sigma_3\tilde{q}^+$$

and is therefore isospectral (up to zero modes of $Q^\pm$) to $H^{(0)}$.

Example with hidden symmetry. Among many possible examples we consider the simple peculiar case when the resulting $\widetilde{H}^{(1)}$ coincides with the initial $H^{(0)}$ up to a constant:

$$H^{(0)} = \widetilde{H}^{(1)} + C. \tag{30}$$

This can be achieved by imposing (see (29)) $G(z) \equiv \mathrm{i}az$ and

$$W_2(z, z^\star) = \frac{1}{2}\left(a\rho + \frac{C}{\rho}\right)\cos\phi + \Theta(\rho)\sin\phi$$

$$W_1(z, z^\star) = \frac{1}{2}\left(a\rho + \frac{C}{\rho}\right)\sin\phi + \Theta(\rho)\cos\phi$$

[9] From now on, we will refer to such global unitary transformation as 'rotation' even disregarding i.

where ρ, ϕ are polar coordinates, a is a constant and $\Theta(\rho)$ is an arbitrary function. The Hamiltonian (30) then becomes

$$H^{(0)} = -\Delta^{(2)} + \frac{1}{4}\left(a\rho + \frac{C}{\rho}\right)^2 + (\Theta(\rho))^2 - a\sigma_1 - \frac{1}{\rho}(\rho\Theta(\rho))'\sigma_3. \tag{31}$$

We stress that though the potential in (31) depends on ρ only, its matrix structure in general prevents a standard separation of variables, typical of scalar problems. From the intertwining (now commutation) relations $[H^{(0)}, Q^{\pm}] = 0$ one can find two mutually commuting Hermitian symmetry operators,

$$R^- \equiv \frac{1}{2\mathrm{i}}(Q^+ - Q^-) = \mathrm{i}a\partial_\phi \tag{32}$$

$$R^+ \equiv \frac{1}{2}(Q^+ + Q^-) = -\sigma_3 H^{(0)} + \mathrm{i}a\sigma_2\rho\partial_\rho + \sigma_3\frac{1}{2}a\rho\left(a\rho + \frac{C}{\rho}\right) - \sigma_1 a\rho\Theta(\rho) \tag{33}$$

with column-eigenfunctions, common to $H^{(0)}$, $R^{\pm}$:

$$\mathrm{e}^{\mathrm{i}m\phi}\hat{\psi}_m(\rho).$$

The first one, R^-, reflects the obvious symmetry of $H^{(0)}$ under rotations, but the second symmetry operator (33) realizes certain 'hidden' symmetry and has the property

$$(R^+)^2 + (R^-)^2 = H^{(0)}(H^{(0)} - C).$$

One can note that the spectrum of the Hamiltonian involves double $m \leftrightarrow -m$ degeneracy, and in addition a possible double degeneracy (for each value of m) associated with a column-function $\hat{\psi}_m(\rho)$. A pedagogical example illustrating the considerations above is obtained from (31) with $\Theta(\rho) \equiv 0$ and $C = 0$: it is exactly solvable after suitable rotation and leads to a pair of decoupled radial oscillators with frequency a and a relative ground state energy shift $2a$.

Higher order iterations. In general, the procedure of gluing with a global rotation enlarges essentially the class of Hamiltonians to which our method is applicable. We now demonstrate this considering two options to construct third-order transformations:

(1) The first option reads

$$H^{(0)} \longrightarrow H^{(1)} \equiv \widetilde{H}^{(0)} + C \longrightarrow \widetilde{H}^{(1)} + C \equiv \sigma_3\widetilde{\widetilde{H}}^{(0)}\sigma_3 + C + \widetilde{C} \longrightarrow \widetilde{\widetilde{H}}^{(1)} + C + \widetilde{C}$$

where '$\longrightarrow$' denotes the appropriate first-order matrix SUSY transformation induced by $q^+, \tilde{q}^+, \tilde{\tilde{q}}^+$, and '$\equiv$' represents the gluing. The first gluing is of the kind described in section 2 (equation (10)), and the superpotentials $W(z, z^\star)$ and $\widetilde{W}(z, z^\star)$ satisfy equations (17) and (18). The second gluing is described in equation (23), and $\widetilde{W}(z, z^\star)$, $\widetilde{\widetilde{W}}(z, z^\star)$ satisfy equations (27) and (28).

Equation (28) can be rewritten as

$$(G(z)\partial F^\star + G^\star(z^\star)\bar{\partial}F) - |G|^2 - \widetilde{C} = 0 \tag{34}$$

where $F(z, z^\star)$ still satisfies (18). We now restrict ourselves to the case $C = 0$, for which equation (18) has trivial particular solutions with purely real or purely imaginary F. Then, for example, for real F, equation (34) can be solved by replacing $z, z^\star$ by

$$t \equiv \int \frac{\mathrm{d}z}{2G(z)} + \int \frac{\mathrm{d}z^\star}{2G^\star(z^\star)} \qquad \tau \equiv \mathrm{i}\left(\int \frac{\mathrm{d}z}{2G(z)} - \int \frac{\mathrm{d}z^\star}{2G^\star(z^\star)}\right) \tag{35}$$

since equation (34) in new variables reads

$$\partial_t F_1 = |G|^2 + \widetilde{C}.$$

A solution for F_1,

$$F_1 = T(\tau) + \int |G|^2\mathrm{d}t + \widetilde{C}t$$

depends on two arbitrary functions—one holomorphic ($G(z)$) and one real ($T(\tau)$). The $F = F_1$ so obtained generates W and $\widetilde{W}$ when inserted into (17) and (27) (in terms of the initial variables x_1, x_2). Thus one obtains a class of Hamiltonians $H^{(0)}$, $(\widetilde{\widetilde{H}}^{(1)} + C + \widetilde{C})$, which are intertwined (and are isospectral) by the third-order operator $Q^+ = q^+\tilde{q}^+\sigma_3\tilde{\tilde{q}}^+$.

(2) The second option includes two gluings, both with σ_3-rotations:

$$H^{(0)} \longrightarrow H^{(1)} \equiv \sigma_3\widetilde{H}^{(0)}\sigma_3 + C \longrightarrow \widetilde{H}^{(1)} + C \equiv \sigma_3\widetilde{\widetilde{H}}^{(0)}\sigma_3 + C + \widetilde{C} \longrightarrow \widetilde{\widetilde{H}}^{(1)} + C + \widetilde{C}.$$

The superpotentials $W(z, z^\star)$ and $\widetilde{W}(z, z^\star)$ have to be found from equations of the form (28), and then $\widetilde{W}(z, z^\star)$ and $\widetilde{\widetilde{W}}(z, z^\star)$ from equations of the form (27). One can obtain the general solution of these equations purely algebraically in terms of two arbitrary holomorphic functions $G(z)$, $\widetilde{G}(z)$,

$$W(z, z^\star) = \frac{G^2\widetilde{G} - |\widetilde{G}|^2 G - \widetilde{C}G - C\widetilde{G}^\star}{\widetilde{G}G - \widetilde{G}^\star G^\star} \qquad (36)$$

with $\widetilde{W}(z, z^\star)$, $\widetilde{\widetilde{W}}(z, z^\star)$ being derived, again algebraically, from two equations of the form (27). Thus one obtains a new class of Hamiltonians $H^{(0)}$, $(\widetilde{\widetilde{H}}^{(1)} + C + \widetilde{C})$, which are intertwined by the third-order operator $Q^+ = q^+\sigma_3\tilde{q}^+\sigma_3\tilde{\tilde{q}}^+$, and are therefore isospectral up to zero modes of $Q^\pm$.

4. Second-order supersymmetrization of scalar two-dimensional Hamiltonians

In this concluding section we will discuss particular cases of sections 2 and 3, when the initial Hamiltonian $H^{(0)}$ is constrained to be diagonal. The condition of diagonality of (7) reads

$$\partial_2 W_1 - \partial_1 W_2 = 0 \quad \Leftrightarrow \quad W_2 = \partial_2\chi(x_1, x_2) \qquad W_1 = \partial_1\chi(x_1, x_2) \qquad (37)$$

where $\chi(x_1, x_2)$ is an arbitrary *real* function. In terms of this function one has $H^{(0)} = \mathrm{diag}\big(H_{11}^{(0)}, H_{22}^{(0)}\big)$ with

$$H_{11}^{(0)} = -\Delta^{(2)} + (\partial_i\chi)^2 - \big(\partial_1^2 + \partial_2^2\big)\chi \qquad H_{22}^{(0)} = H_{11}^{(0)} + 2\big(\partial_1^2 + \partial_2^2\big)\chi \qquad (38)$$

which is intertwined (and therefore, isospectral, up to zero modes of supercharges (9)) with non-diagonal Hamiltonian $H^{(1)}$. The latter is obtained by inserting (37) into (8). From now on we study the iteration of SUSY transformations for the initial Hamiltonian (38), i.e., the compatibility of conditions (37) with the iteration algorithm.

Simple gluing. For the case $C = 0$ in (10) the combination of equation (17) with equation (37) can be solved in terms of an arbitrary real function $\xi(x_1, x_2)$:

$$F_1(x_1, x_2) = \big(\partial_2^2 - \partial_1^2\big)\xi(x_1, x_2) \qquad (39)$$

$$F_2(x_1, x_2) = -2\partial_1\partial_2\xi(x_1, x_2) \qquad (40)$$

$$\chi(x_1, x_2) = \big(\partial_1^2 + \partial_2^2\big)\xi(x_1, x_2). \qquad (41)$$

The last condition to be taken into account is the interrelation (19) between F_1 and F_2 :

$$\big(\partial_2^2 - \partial_1^2\big)\xi(x_1, x_2) = f[\partial_1\partial_2\xi(x_1, x_2)]. \qquad (42)$$

Of course, one cannot find the general solution of this nonlinear functional-differential equation, but one can find a variety of particular solutions. The simplest choice $f = \mathrm{const}$ in (42) is trivial, since the corresponding Hamiltonian $H^{(1)}$ can be directly diagonalized by rotation. Some less trivial solutions of (39), (40), (42) can be found by requiring that the lhs of equations (39) and (40) are, e.g., $F_i = \lambda_i\xi^k$, $F_i = \exp(\lambda_i\xi)$, etc.

For the case $C \neq 0$ one can try similar techniques as used for the solution of equation (18) to solve now equations (18) and (37) together with (17). The resulting equation reads

$$\left(\partial_1^2 - \partial_2^2\right)F_2 = 2\partial_1\partial_2 F_1.$$

Particular solutions of this equation can be found by suitable ansätze like, for example,

$$F_1(x_1, x_2) \equiv x_1 g(x_2) + f_2(x_2) \qquad \Phi(F_1) \equiv aF_1^2 + bF_1 + c + \frac{d}{F_1}$$

where $g(x_2)$, $f(x_2)$ can be determined by solvability conditions, the functional $\Phi(F_1)$ has been introduced in (20), and a, b, c, d are constants. One can also enlarge the class of solutions by suitable nonsingular linear transformations of x_1, x_2.

Gluing with rotation. In this case the nonlinear equation (28) takes the form

$$(\partial_1\chi)G_1 + (\partial_2\chi)G_2 = \tfrac{1}{2}\left[G_1^2 + G_2^2 + C\right]$$

or, in terms of variables $z, z^\star$,

$$(G(z)\partial + G^\star(z^\star)\bar{\partial})\chi(z, z^\star) = |G(z)|^2 + \tfrac{1}{4}C. \tag{43}$$

It has the form similar to equation (34) of the previous section except for the fact that $\chi(z, z^\star)$ is a real function. Therefore, the general solution of (43) is expressed in terms of t, τ of (35) via an arbitrary (real) function $\Lambda(\tau)$:

$$\chi(z, z^\star) = \Lambda(\tau) + \int |G(z)|^2\,\mathrm{d}t + \frac{1}{4}Ct. \tag{44}$$

The initial diagonal Hamiltonian $H^{(0)}$ from (7) and (37) reads

$$\begin{aligned} H^{(0)} &= -\Delta^{(2)} + 4(\partial\chi)(\bar{\partial}\chi) - 4(\partial\bar{\partial}\chi)\sigma_3 \\ &= -\Delta^{(2)} + |G|^2 + \frac{1}{2}C + \frac{C^2}{16|G|^2} + \frac{\Lambda'^2(\tau)}{|G|^2} \\ &\quad + \frac{2\Lambda'(\tau)}{|G|^2}\int \partial_\tau|G|^2\,\mathrm{d}t + \frac{1}{|G|^2}\left(\int \partial_\tau|G|^2\,\mathrm{d}t\right)^2 - \sigma_3\left(G' + G^{\star\prime} + \frac{\Lambda''}{|G|^2}\right) \end{aligned}$$

while the final $\widetilde{H}^{(1)}$ can be expressed as

$$\widetilde{H}^{(1)} + C = H^{(0)} - 2\sigma_1\partial_1 G_2 + 2\sigma_3\partial_1 G_1.$$

As a particular example one can consider the case of $G = z^{1/2}$ which leads to a confining singular Hamiltonian with a singularity $1/\rho$ and growing asymptotically as ρ with a non-trivial azimuthal dependence.

Finally, we note that conversely the intertwining between diagonal $H^{(0)}$ and non-diagonal $\widetilde{H}^{(1)}$ by second-order matrix supercharges can also be used to diagonalize a class of matrix Schrödinger operators (or system of differential equations of second order with non-diagonal matrix coupling), in analogy with similar procedures already used in [13, 14, 20].

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